DTP-95/92
IC/95/353
October, 1995

CONSTRAINT ON THE QED VERTEX FROM THE MASS ANOMALOUS DIMENSION $\gamma_m = 1$

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ABSTRACT

We discuss the structure of the non-perturbative fermion-boson vertex in quenched QED. We show that it is possible to construct a vertex which not only ensures that the fermion propagator is multiplicatively renormalizable, obeys the appropriate Ward-Takahashi identity, reproduces perturbation theory for weak couplings and guarantees that the critical coupling at which the mass is dynamically generated is gauge independent but also makes sure that the value for the anomalous dimension for the mass function is strictly 1, as Holdom and Mahanta have proposed.

In a recent paper [1], we presented a mechanism for constructing an effective non-perturbative vertex in quenched QED which incorporates some of the key features required for a gauge theory. It ensures the fermion propagator is multiplicatively renormalizable, the Ward-Takahashi identity relating the fermion propagator to the fermion-boson vertex is satisfied, reproduces perturbation theory for low values of the coupling and yields a strictly gauge independent critical coupling for dynamical mass generation. This construction builds on the results of Dong et al. [2]. The non-perturbative vertex is written in terms of two unknown functions W_1 and W_2 which obey certain conditions, Eqs. (28,33,46,59) of [1]. With the fermion propagator of momentum p given by

$$S_F(p) \;=\; \frac{F(p^2)}{\not{p} - \mathcal{M}(p^2)} \qquad , \tag{1}$$

the function W_1 corresponds to the equation for the fermion wavefunction renormalization $F(p^2)$, Eq. (12), while W_2 is related to the mass function $\mathcal{M}(p^2)$, Eq. (13) of [1]. One of the assumptions made in that work was that the transverse vertex defined by Eqs. (9,10) of [1] vanishes in the Landau gauge as it does in the leading logarithm approximation. Although this assumption does not enter the discussion of W_1, the conditions for W_2, Eq. (46,59) of [1], crucially depend on it. This issue is intimately related to the value of the anomalous dimension for the fermion mass function, γ_m. In the quenched theory the ultraviolet behaviour of $\mathcal{M}(p^2)$ can be expressed as

$$\mathcal{M}(p^2) \sim (p^2)^{\gamma_m/2-1} \tag{2}$$

in the deep Euclidean region. At criticality, where there is only one momentum scale, Λ the ultraviolet cutoff [3], the mass function behaves as in Eq. (2) at all momenta. If the aforementioned assumption about the vanishing Landau gauge transverse vertex holds true, then $\gamma_m = 1.058$.

However, Holdom [4], followed by Mahanta [5], using arguments based on the Cornwall-Jackiw-Tomboulis (CJT) effective potential technique has shown that γ_m is strictly equal to 1 regardless of the choice of the vertex. If this were true, this would suggest that there is a necessary piece in the transverse part of the effective vertex which does not vanish

in the Landau gauge. A recent perturbative calculation of the transverse vertex in an arbitrary covariant gauge, performed by Kızılersü et al. [6], reveals that the transverse part of the actual vertex *does not* vanish in the Landau gauge. This fact may possibly favour Holdom's conclusions. It may well be that the non-zero transverse piece in the Landau gauge restores the simplicity of the result which is the characteristic of the bare vertex, spoiled by an additional term introduced in the longitudinal vertex proposed by Ball and Chiu [7]. In this comment, we show that one could construct a vertex which ensures that the gauge invariant chiral symmetry breaking takes place with $\gamma_m = 1$ instead of $\gamma_m = 1.058$. This vertex again involves the same function W_1, but instead of W_2, we obtain a function V_2 which obeys slightly different conditions compared to Eqs. (46,59) of [1]. We use the same definitions and notations as in [1] unless mentioned otherwise.

Firstly, we recall that if the equation for $F(p^2)$ is to have a solution that is multiplicatively renormalizable, then it must behave as

$$F(p^2) = \left(\frac{p^2}{\Lambda^2}\right)^{\nu} \quad , \tag{3}$$

where $\nu = \alpha\xi/4\pi$ in keeping with the Landau-Khalatnikov transformation [8].

It is then well known that in the case of the bare vertex, the mass function obeys the following linearized equation in Euclidean space in the Landau gauge, where $F(p^2) = 1$:

$$\mathcal{M}(p^2) = \frac{3\alpha}{4\pi} \int_0^{\Lambda^2} \frac{dk^2}{k^2} \mathcal{M}(k^2) \left[\frac{k^2}{p^2}\theta(p^2 - k^2) + \theta(k^2 - p^2)\right] \quad . \tag{4}$$

This equation has the multiplicatively renormalizable solution of the form of Eq. (2), with

$$\gamma_m = 1 \pm \sqrt{1 - \frac{\alpha}{\alpha_c}} \quad , \tag{5}$$

where $\alpha_c = \pi/3$. When $\alpha = \alpha_c$, $\gamma_m = 1$. (Note that in [1] the exponent in Eq. (2) is called $-s$, so that $\gamma_m = 2(1 - s)$.) In order that Eq. (34) of [1] is identical to Eq. (4) for all values of the covariant gauge parameter, ξ, the following must hold true :

$$
\begin{aligned}
& \frac{\xi}{3}\int_0^{p^2}\frac{dk^2}{p^2}\,\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)} \\
= \; & -\int_0^{p^2}\frac{dk^2}{p^2}\frac{1}{2(k^2-p^2)}\left[p^2\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)-k^2\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right] \\
& -\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{1}{2(k^2-p^2)}\left[k^2\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)-p^2\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right] \\
& +\int_0^{p^2}\frac{dk^2}{p^2}\mathcal{M}(k^2)F(k^2)\left[\frac{k^2}{6}(k^2-3p^2)\,\tau_2(k^2,p^2)+p^2\,\tau_3(k^2,p^2)\;+\;(k^2-p^2)\,\tau_6(k^2,p^2)\right] \\
& +\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\mathcal{M}(k^2)F(k^2)\left[\frac{p^2}{6}(p^2-3k^2)\,\tau_2(k^2,p^2)+k^2\,\tau_3(k^2,p^2)\;+\;(k^2-p^2)\,\tau_6(k^2,p^2)\right],
\end{aligned}
\tag{6}
$$

where the τ_i, $i=2,3,6,8$ (τ_8 only occurs in the analogous equation for $F(p^2)$) are the coefficients of the transverse vertex in the basis of Ball and Chiu [7]. Demanding that a chirally symmetric solution should be possible when the bare mass is zero is most easily accomplished if only those transverse vectors with odd numbers of gamma matrices contribute to the transverse vertex. That is why, τ_i, $i=1,4,5,7$, have dropped out.

Introducing in Eq. (6) the variable x, where, for $0\le k^2<p^2$, $x=k^2/p^2$, and for $p^2\le k^2<\Lambda^2$, $x=p^2/k^2$, we can now retrace the steps carried out in obtaining Eq. (46) in [1], starting from Eq. (41). This will lead us to the compact equation

$$
\int_0^1\frac{dx}{\sqrt{x}}\,V_2(x)=0 \qquad , \tag{7}
$$

where using Eq. (3)

$$
\begin{aligned}
V_2(x) \;=\; & \xi\, x^{\nu}+\frac{3}{2}\left[\frac{x^{-\nu}-x^{\nu}}{x-1}\right]-\frac{3x}{2}\left[\frac{x^{-(\nu+\frac{1}{2})}-x^{(\nu+\frac{1}{2})}}{x-1}\right] \\
& -x^{\nu}\left[g_1(x)+g_2(x)\right]-x^{-\nu}\left[g_1(1/x)-g_2(1/x)\right]\;.
\end{aligned}
\tag{8}
$$

The functions $g_1(x)$ and $g_2(x)$ are as defined in [1]. Here, the function V_2 is the counterpart of W_2 in [1]. Eqs. (7,8) should be compared with Eqs. (46,47) of [1]. The condition Eq. (7) ensures that $\gamma_m=1$ in any covariant gauge, just as condition Eq. (46) ensures $\gamma_m=1.058$. As expected, unlike W_2, V_2 does not vanish in the Landau gauge. Instead, it is

$$
V_2(x)=\frac{3\sqrt{x}}{2}\;-\;2\left[g_1(x)+g_2(x)\right] \qquad . \tag{9}
$$

Eq. (8) can be inverted to evaluate the expressions for τ_i in terms of V_2. We shall not give the expression for τ_6 as it is solely related to the equation for the wavefunction renormalization, Eq. (14) of [1], and hence remains completely independent of the value of γ_m. Repeating the same steps as in [1], we obtain

$$\begin{aligned}
\tau_2(k^2,p^2) &= -6\,\frac{\tau_6(k^2,p^2)}{(k^2-p^2)} \\
&+\frac{1}{(k^2-p^2)^2}\frac{1}{[F(k^2)+F(p^2)]}\left\{3(k^2+p^2)R_2(k^2,p^2)+2\xi Q_2(k^2,p^2)\right\} \\
&-\frac{1}{(k^2-p^2)^2}\;\frac{1}{[F(k^2)+F(p^2)]}\left[V_2\left(\frac{k^2}{p^2}\right)+V_2\left(\frac{p^2}{k^2}\right)\right] \\
&-\frac{k^2+p^2}{(k^2-p^2)^3}\;\frac{1}{[F(k^2)+F(p^2)]}\left[V_2\left(\frac{k^2}{p^2}\right)-V_2\left(\frac{p^2}{k^2}\right)\right],
\end{aligned} \tag{10}$$

where both $Q_2(k^2,p^2)$ and $R_2(k^2,p^2)$ are symmetric functions of k and p :

$$\begin{aligned}
Q_2(k^2,p^2) &= \frac{1}{k^2-p^2}\left[k^2\,\frac{F(k^2)}{F(p^2)}-p^2\,\frac{F(p^2)}{F(k^2)}\right] \\
R_2(k^2,p^2) &= \frac{1}{k^2-p^2}\left[\frac{F(k^2)\mathcal{M}(p^2)}{F(p^2)\mathcal{M}(k^2)}-\frac{F(p^2)\mathcal{M}(k^2)}{F(k^2)\mathcal{M}(p^2)}\right],
\end{aligned} \tag{11}$$

$$\begin{aligned}
\tau_3(k^2,p^2) &= -\frac{k^2+p^2}{k^2-p^2}\,\tau_6(k^2,p^2) \\
&+\frac{1}{(k^2-p^2)^2}\frac{1}{[F(k^2)+F(p^2)]}\left\{2k^2p^2R_2(k^2,p^2)-\frac{\xi}{3}Q_3(k^2,p^2)\right\} \\
&-\frac{1}{6}\frac{k^2+p^2}{(k^2-p^2)^2}\;\frac{1}{[F(k^2)+F(p^2)]}\left[V_2\left(\frac{k^2}{p^2}\right)+V_2\left(\frac{p^2}{k^2}\right)\right] \\
&+\frac{1}{6}\frac{k^4+p^4-6k^2p^2}{(k^2-p^2)^3}\;\frac{1}{[F(k^2)+F(p^2)]}\left[V_2\left(\frac{k^2}{p^2}\right)-V_2\left(\frac{p^2}{k^2}\right)\right],
\end{aligned} \tag{12}$$

where

$$Q_3(k^2,p^2) = \frac{1}{k^2-p^2}\left[p^2(p^2-3k^2)\frac{F(k^2)}{F(p^2)}-k^2(k^2-3p^2)\frac{F(p^2)}{F(k^2)}\right] \quad . \tag{13}$$

and

$$
\begin{aligned}
\tau_8(k^2,p^2) &= -2\,\frac{k^2+p^2}{k^2-p^2}\,\tau_6(k^2,p^2)+\overline{\tau}(k^2,p^2) \\
&\quad +\frac{1}{(k^2-p^2)^2}\frac{1}{[F(k^2)+F(p^2)]} \\
&\qquad \left\{\frac{1}{2}(3k^4+3p^4+2k^2p^2)R_2(k^2,p^2)+\frac{\xi}{3}Q_8(k^2,p^2)\right\} \\
&\quad -\frac{1}{3}\,\frac{k^2+p^2}{(k^2-p^2)^2}\,\frac{1}{[F(k^2)+F(p^2)]}\left[V_2\left(\frac{k^2}{p^2}\right)+V_2\left(\frac{p^2}{k^2}\right)\right] \\
&\quad -\frac{2}{3}\,\frac{k^4+p^4}{(k^2-p^2)^3}\,\frac{1}{[F(k^2)+F(p^2)]}\left[V_2\left(\frac{k^2}{p^2}\right)-V_2\left(\frac{p^2}{k^2}\right)\right],
\end{aligned}
\tag{14}
$$

where

$$
Q_8(k^2,p^2) = \frac{1}{(k^2-p^2)}\left[(3k^4+p^4)\frac{F(k^2)}{F(p^2)}-(k^4+3p^4)\frac{F(p^2)}{F(k^2)}\right] \tag{15}
$$

and $\overline{\tau}(k^2,p^2)$ is defined by Eq. (17) of [1]. Note that all the momenta above are in Euclidean space. Therefore, appropriate changes of sign have to be made in order to get the expressions for τ_i in the Minkowski space to construct the transverse vertex using the basis vectors of Ball and Chiu [7]. Eqs. (10–15) should be compared with Eqs. (52–58) of [1]. It is here that we note the restoration of simplicity. In contrast with $q_i(k^2,p^2)$, $Q_i(k^2,p^2)$ do not have any dependence on the mass functon $\mathcal{M}(p^2)$ at all. Moreover, the explicit appearance of the mass term in τ_2, τ_3 and τ_8 in the present case is only through the factor

$$
\frac{1}{k^2-p^2}\left[\frac{F(k^2)\mathcal{M}(p^2)}{F(p^2)\mathcal{M}(k^2)}-\frac{F(p^2)\mathcal{M}(k^2)}{F(k^2)\mathcal{M}(p^2)}\right],
$$

unlike the case with $\gamma_m = 1.058$, where $r_2(k^2,p^2)$, $q_2(k^2,p^2)$, $q_3(k^2,p^2)$ and $q_8(k^2,p^2)$ all carry the dependence on the mass function $\mathcal{M}(p^2)$ through different and more complicated terms.

Imposing the condition that the vertex and its components should be free of kinematic singularities now implies,

$$
V_2(1)+2V_2'(1) = 2\nu(\xi+3)+(\xi+6-3\gamma_m) \qquad , \tag{16}
$$

which replaces Eq. (59) of [1] and at the critical coupling, it reduces to $V_2(1) + 2V_2'(1) = (\xi+3)(2\nu+1)$. The transverse vertex now has the correct lowest order perturbative limit, viz. $\Gamma_T^\mu = \mathcal{O}(\alpha)$, provided

$$V_2\left(k^2/p^2\right) = \xi\,\frac{F(k^2)}{F(p^2)} + \frac{3}{2}\left[\frac{F(k^2)\mathcal{M}(p^2)}{F(p^2)\mathcal{M}(k^2)} - \frac{F(p^2)\mathcal{M}(k^2)}{F(k^2)\mathcal{M}(p^2)}\right] + \mathcal{O}(\alpha) \quad . \tag{17}$$

Note that Eq. (7) is only true at the bifurcation point just as Eq. (46) in [1] whose exact form for all α might be suggested by Eq. (5) to be

$$\int_0^1 \frac{dx}{\sqrt{x}}\, V_2(x) \approx 2\,\xi\,\sqrt{1-\frac{\alpha}{\alpha_c}} \quad . \tag{18}$$

in order to agree with both the $\alpha = 0$ and $\alpha = \alpha_c$ limits, Eqs. (17,7).

We have been able to show that there is no technical difference between the mechanism of constructing the transverse vertex for the case $\gamma_m = 1$ and $\gamma_m = 1.058$. On comparing Eq. (47) of [1] and Eq. (8), we can see that the main difference between V_2 and W_2 is that V_2 has an additional piece coming from the longitudinal part of the vertex. As a result of this difference, V_2 does not vanish in the Landau gauge in contrast to W_2. It is this that ensures at criticality that the anomalous dimension $\gamma_m = 1$ identically in all covariant gauges in keeping with the results of [4, 5].

Acknowledgements

AB wishes to thank the Government of Pakistan for a research studentship and International Centre for Theoretical Physics (ICTP), Trieste, for their hospitality and support during his visit to the Centre.

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